EXISTENCE OF NONDETERMINED SETS FOR SOME TWO PERSON GAMES OVER REALS

BY

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ABSTRACT

Various two person games with perfect information over reals are shown to have a nondetermined set. A game formulated by Mycielski is solved.

1. Introduction

Let X be a set, P a subset of X^{ω}. $G_X(P)$ denotes the following two person game [1]:

players I and II choose alternately $x_0, x_1, \dots, x_i \in X$; player I starts.

I wins iff the resulting sequence belongs to P.

P is *determined for* G iff one of the players has a winning strategy.

Instances of this game for $X = R$ (the set of real numbers) were first described probably, in the so-called Scottish Book [12] by Banach and Mazur, and subsequently studied and modified by different authors ([1]–[9]).

These games are usually not formulated by specifying a $P \subseteq R^{\omega}$, but rather deal with some sort of constructive process, performed by the players, that might "lead" to a real number (e.g. by constructing a converging sequence). Winning conditions are then defined in terms of subsets of R rather than in terms of subsets of R^{ω}; namely, I wins iff the resulting real belongs to some $S \subseteq R$. We then speak of $\Gamma(S)$ rather than of the corresponding $G_R(P)$. *S is determined for* Γ if one of the players has a winning strategy.

Mycielski ([9], § 4.2. (d)) raised the question, whether nondetermined sets exist for such Γ's. We show that the answer for this question is positive, for all games Γ so formulated, known to us, if the existence of a well-ordering of the

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continuum is assumed. More precisely, all those modifications Γ share the property, that if S is a subset of R such that neither S nor its complement includes a perfect subset, then S is nondetermined for Γ. This result is formulated as Theorem 1. After having proved this theorem, the author was informed by J. Mycielski that the case of Γ_1 (in the terminology of that theorem) follows from an unpublished result of R. Solovay.

In §4, a solution is given for a game formulated by Mycielski in [9], which we denote as Γ_1. It is shown that player I wins $\Gamma(S)$ iff S includes a perfect subset, and player II wins $\Gamma(S)$ iff S is at most denumberable. This result is formulated in §2 as Theorem 2. In this formulation it was proved by R. Solovay in 1968, but not published. The author wishes to express his thanks to J. Mycielski for calling his attention to Solovay's result, and to R. Solovay for his kind permission to publish it.

The existence of an S such that neither S nor its complement includes a perfect subset follows from the existence of a well ordering of the continuum. Thus, the existence of nondetermined S for those Γ's is a theorem of ZFC. Solovay [10] showed that if one assumes the existence of an inaccessible cardinal, one can consistently assume (in ZF) that every infinite subset of R is either countable or contains a perfect subset. Hence, under this assumption, it is consistent to assume that Mycielski's game is determined.

2. Formulation of the games and statement of the results.

Players I and II choose alternately $x_0, x_1, \ldots, x_i \in R$. I starts. S is a subset of R. $\Gamma_i(S)$ denotes the game subject to the condition i, $i = 1,2,3$, as follows:

$i = 1$: Game rules; $x_{2n+1} < x_{2n+3} < x_{2n+2} < x_{2n}$, $n = 0, 1, \cdots$.
Winning condition: I wins iff $\lim_{n\to\infty} x_{2n} \in S$

$i = 2$: Game rules: $|x_{n+1} - x_n| \leqq 2^{-n}$ $n = 0,1,\cdots$.
Winning condition: I wins iff $\lim_{n\to\infty} x_n \in S$

$i = 3$: two sequences of positive numbers k_n, l_n are given.
Game rules: $0 < x_{2n} < l_{n-1}x_{2n-1}$ $n = 1,2,\cdots$
$0 < x_{2n+1} < k_n x_{2n}$ $n = 0,1,2,\cdots$
Winning condition: I wins iff Σx_n converges and belongs to S.

$\Gamma_1(S)$ was formulated by Mycielski in [9] as an easy modification of his Example 3 in [3]. All the results stated here about the former hold for the latter.

$\Gamma_2(S)$ is Example 2 of [3] (Mycielski).

$\Gamma_3(S)$ is a generalizaton of formulations by Banach and Mazur ([12], where $k_n = l_n = 1$), by H. Hanani and M. Reichbach's in [7] (where $k_n = 1$ and $l_n = l$ is constant) and by H. Hanani's in [6] (where $k_n = 1$ and l_n is a non-decreasing sequence). Example 1 in [3] is obtained from $\Gamma_3(S)$ by puting $k_n = l_n = 1$ and replacing the right $<$ by $\leqq$; our proof covers—with the necessary changes—this case as well.

THEOREM 1. *If neither S nor its complement includes a perfect subset, then* $\Gamma_i(S)$ *is nondetermined*, $i = 1,2,3$.

THEOREM 2. $\Gamma_1(S)$ *is a win for* I iff *S includes a perfect subset, and it is a win for* II *iff S is at most denumerable.*

Theorems 1 and 2 are proved in ZF, with no use of the axiom of choice. As we already mentioned, using the assumption that the continuum can be well ordered — which is a theorem if we add the axiom of choice to ZF — one can prove the existence of a set $S \subseteq R$ such that neither S nor its complement includes a perfect subset. Thus we have, in ZFC:

There exist a nondetermined S for Γ_i, $i = 1,2,3$.

By Theorem 2 and Solovay's result [10], mentioned in the introduction, a proof for this fact that makes no use of the axiom of choice for $i = 1$ is impossible (unless ZF + "there exist an inaccessible cardinal" is inconsistent).

3. Proof of Theorem 1

Detailed proofs of the theorem for $i = 1,2,3$ are different for technical reasons, but the argument is common, thus we first describe it informally:

A strategy of a player for one of these games is a function from R^*, the set of all finite sequences of elements of R, into R, which is subject to the game rules. The finite sequence is interpreted as a sequence of moves taken hitherto by his opponent.

We show that, considering all plays in which a player adheres to a specified strategy, one can single out a collection of outcomes that forms a perfect subset.

The symmetry of the rules of the considered games enables us to do with just "one half" of the proof each time, showing only that the set of results of plays of II against a given strategy of I includes a perfect subset. The reader should find no difficulty in proving similarly, that the set of results of plays of I against a given strategy of II includes a perfect subset as well.

We conclude that if I has a winning strategy, S should include a perfect subset, and that if II has a winning strategy, the complement of S should include a perfect subset. This establishes the theorem.

NOTATION. For a set A, A^* denotes the set of all finite sequences of elements from A. A^* contains also the empty sequence $\emptyset$. For $\xi = \langle x_0,\cdots,x_{n-1}\rangle \in A^*$, we put $|\xi| = n$, and for $k \leqq n, \bar{\xi}(k) = \langle x_0,\cdots,x_{k-1}\rangle$. $\xi\cdot\eta$ denotes the concatenation of ξ and η.

A^ω denotes the set of all ω-sequences over A. For $\alpha \in A^\omega$, $\bar{\alpha}: \omega \to A^*$ is defined by

$$\bar{\alpha}(n) = \langle\alpha(0),\cdots,\alpha(n-1)\rangle$$

For a function $\gamma: A^* \to B$, $\tilde{\gamma}: A^* \to B^*$ is defined by

$$\tilde{\gamma}(\langle a_0,\cdots,a_{n-1}\rangle) = \langle\gamma(\langle a_0\rangle), \gamma(\langle a_0,a_1\rangle),\cdots, \gamma(\langle a_0,\cdots,a_{n-1}\rangle)\rangle$$

"Perfect set" will always mean "perfect nonempty subset of R". 2 denotes the set $\{0,1\}$. We need the well known

LEMMA. *If P_u is a closed nonempty interval for all $u \in 2^*$, and*

(1) $P_{u\cdot\langle 0\rangle} \subset P_u$, $P_{u\cdot\langle 1\rangle} \subset P_u$

(2) $P_{u\cdot\langle 0\rangle} \cap P_{u\cdot\langle 1\rangle} = \emptyset$

then

$$P = \bigcap_{n<\omega} \bigcup_{|u|=n} P_u$$

is a perfect set.

PROOF FOR $i = 1$.

Let $\sigma: R^* \to R$.

$\xi = \langle x_0,\cdots,x_{n-1}\rangle \in R^*$ is a *σ-sequence* iff: $n = 0$ or $n > 0$ and

$$x_0 < \cdots < x_{n-1} < \sigma(\bar{\xi}(n-1)) < \cdots < \sigma(\emptyset)$$

σ is a *stratgy of I* iff for all $\xi \in R^*$, if ξ is a σ-sequence, $|\xi| = n > 0$, then

$$x_{n-1} < \sigma(\xi) < \sigma(\bar{\xi}(n-1))$$

If σ is a strategy of I, $\alpha \in R^\omega$, we say that α *is good for* σ if for all $n < \omega$, $\bar{\alpha}(n)$ is a σ-sequence.

For $\alpha \in R^\omega$, denote by $\sigma{*}{*}\alpha \in R^\omega$ the sequence of real numbers defined by:

$$\sigma{*}{*}\alpha(n) = \sigma(\bar{\alpha}(n))$$

Finally, define for a strategy σ of I

(1) $A_\sigma = \{\sigma * \alpha : \alpha \text{ is good for } \sigma\}$

where

(2) $\sigma * \alpha = \lim \ \sigma{*}{*}\alpha(n)$

$\sigma * \alpha$ is the outcome of a play where II plays α, against σ (observe that if α is good for σ, then $\sigma * * \alpha$ is a bounded decreasing sequence).

We want to show: A_σ includes a perfect subset.

Define $\beta: 2^* \to R$ by induction on $|u|$, $u \in 2^*$, as follows:

$$\beta(\emptyset) = \sigma(\emptyset) - 1$$

$$\beta(u \cdot \langle 0 \rangle) = \tfrac{1}{2}(\beta(u) + \sigma(\tilde{\beta}(u)))$$

$$\beta(u \cdot \langle 1 \rangle) = \tfrac{1}{2}(\sigma(\tilde{\beta}(u \cdot \langle 0 \rangle)) + \sigma(\tilde{\beta}(u)))$$

It is easy to verify by induction on $|u|$ that:

(3) $\tilde{\beta}(u)$ is a σ-sequence for $u \in 2^*$

(4) $\beta(u) < \beta(u \cdot \langle 0 \rangle) < \sigma(\tilde{\beta}(u \cdot \langle 0 \rangle) < \beta(u \cdot \langle 1 \rangle) < \sigma(\tilde{\beta}(u \cdot \langle 1 \rangle)) < \sigma(\tilde{\beta}(u))$

(5) If u contains k zeros and l ones,

$$\sigma(\tilde{\beta}(u)) - \beta(u) \leqq 2^{-(k+2l)}$$

Define for $u \in 2^*$:

$$P_u = [\beta(u),\ \sigma(\tilde{\beta}(u))]$$

By (4)

$$(6) \quad P_{u \cdot \langle 0 \rangle} \subseteq P_u,\ P_{u \cdot \langle 1 \rangle} \subseteq P \qquad P_{u \cdot \langle 0 \rangle} \cap P_{u \cdot \langle 1 \rangle} = \emptyset$$

Hence

$$P = \bigcap_{n < \omega} \bigcup_{\substack{|u| = n \\ u \in 2^*}} P_u$$

is a perfect set.

Now, by (5) and (6) $x \in P$ iff there is a unique $\delta \in 2^\omega$ such that $x \in P_{\bar{\delta}\langle n \rangle}$ for all $n < \omega$.

Hence, $x \in P$ iff $x = \lim \sigma(\tilde{\beta}(\bar{\delta}(n)))$ for a unique $\delta \in 2^\omega$.

But, by (3), $\beta(\delta) \in R^\omega$ defined by $\beta(\delta)(n) = \beta(\bar{\delta}(n))$ is good for σ.

Hence, $x = \sigma * \beta(\delta) \in A_\sigma$ (by (1), (2)), so

$$P \subseteq A_\sigma$$

This completes the proof for $i = 1$.

Proof for $i = 2$.

Let $\sigma: R^* \to R$

$\xi \in R^*$ is a *σ-sequence* iff $\xi = \emptyset$ or $\xi = \langle x_0, \cdots, x_{n-1} \rangle$, $n > 0$, and

(7) $|x_k - \sigma(\bar{\xi}(k))| \leqq 2^{-2k}$, $0 \leqq k < n$.

σ is a *strategy for I* if for all σ-sequence ξ with $|\xi| = n > 0$ as above

(8) $\left|\sigma(\xi) - x_{n-1}\right| \leqq 2^{-(2(n-1)+1)}$

$\alpha \in R^{\omega}$ is *good for* σ if for all $n < \omega$, $\bar{\alpha}(n)$ is a σ-sequence.

If σ is a strategy for I, α is good for σ, then $\alpha(n)$ is a convergent sequence, and $\lim \alpha(n)$ is the result of the play where I follows σ and II plays α.

Fix a strategy σ for I. For α good for σ, denote as before

$$\sigma * \alpha = \lim \alpha(n)$$

$$A_{\sigma} = \{\sigma * \alpha : \alpha \text{ is good for } \sigma\}$$

We have to show: A_σ includes a perfect set. Observe first, that for α good for σ we have, by (7), (8)

$$\left|\sigma(\bar{\alpha}(k+1)) - \alpha(k)\right| \leqq 2^{-(2k+1)}$$

$$\left|\alpha(k+1) - \sigma(\bar{\alpha}(k+1))\right| \leqq 2^{-2(k+1)}$$

Hence:

$$\left|\alpha(k+1) - \alpha(k)\right| \leqq 2^{-(2k+1)} + 2^{-2(k+1)}$$

and:

$$\left|\sigma * \alpha - \alpha(n)\right| = \left|\sum_{i=0}^{\infty} \alpha(n+i+1) - \alpha(n+i)\right| \leqq \sum_{i=0}^{\infty} \left|\alpha(n+i+1) - \alpha(n+i)\right|$$

$$\leqq \sum_{i=0}^{\infty} (2^{-(2(n+i)+1)} + 2^{-2(n+i+1)}) = 2^{-2n} \sum_{i=1}^{\infty} 2^{-i} = 2^{-2n}$$

Thus

(9) $\left|\sigma * \alpha - \alpha(n)\right| \leqq 2^{-2n}$

Define now inductively the function β: $2^* \to R$, by

$$\beta(\emptyset) = 0$$

and for $u \in 2^*$ with $|u| = n$:

$$\beta(u \cdot \langle 0 \rangle) = \sigma(\tilde{\beta}(u)) - 2^{-2n}$$

$$\beta(u \cdot \langle 1 \rangle) = \sigma(\tilde{\beta}(u)) + 2^{-2n}$$

By induction one shows:

$$\tilde{\beta}(u) \text{ is a } \sigma\text{-sequence for } u \in 2^*.$$

For $u,v,w \in 2^*$ with $|u| = n$ we have

(10) $\beta(u \cdot \langle 0 \rangle \cdot v) < \beta(u \cdot \langle 0 \rangle) + 2^{-2n} = \sigma(\tilde{\beta}(u)) =$

$= \beta(u \cdot \langle 1 \rangle) - 2^{-2n} < \beta(u \cdot \langle 1 \rangle \cdot w).$

For $\delta \in 2^\omega$, define $\beta(\delta) \in R^\omega$ by

$$\beta(\delta)(n) = \beta(\bar{\delta}(n+1))$$

Then $\beta(\delta)$ is good for σ for every $\delta \in 2^\omega$.

Define a function f: $2^\omega \to R$ by

$$f(\delta) = \sigma * \beta(\delta)$$

By definition, $f(2^\omega) \subseteq A_\sigma$. Thus, to complete the proof, we show that $f(2^\omega)$ includes a perfect set.

It is easy to see that f is a continuous mapping of 2^ω into R, where 2^ω is endowed with the product topology induced by the discrete topology on 2. 2^ω with this topology is homeomorphic to Cantor's discontinuum. One proves without making use of the axiom of choice that if f is one-to-one on an uncountable subset of 2^ω, then $f(2^\omega)$ includes a subset homeomorphic to 2^ω (see [11]), i.e. $f(2^\omega)$ includes a perfect set. We show now that such a set exists.

Define $B \subseteq 2^\omega$ by

$$B = \{\delta \in 2^\omega\colon \delta^{-1}(\{1\}) \text{ is infinite}\}$$

B differs from 2^ω by a countable set, namely the set of $\delta \in 2^\omega$ that are ultimately zero. Thus, B has the power of the continuum.

We will show that f is one-to-one on B.

Let: $\delta_1, \delta_2 \in B$, $\delta_1 \neq \delta_2$.

Put: $$n_0 = \min \{k\colon \delta_1(k) \neq \delta_2(k)\}.$$

With no loss of generality, assume $\delta_2(n_0) = 1$, and let:

$$m_0 = \min \{k\colon k > n_0 \text{ and } \delta_2(k) = 1\}$$

By (9) and (10), we now have

$$\begin{aligned}
f(\delta_1) &= \sigma * \beta(\delta_1) \leqq \beta(\delta_1)(n_0) + 2^{-2n_0} = \beta(\bar{\delta}_1(n_0+1)) + 2^{-2n_0} \\
&= \beta((\bar{\delta}_1(n_0))\cdot\langle 0\rangle) + 2^{-2n_0} = \beta((\bar{\delta}_2(n_0))\cdot\langle 0\rangle) + 2^{-2n_0} \\
&= \beta((\bar{\delta}_2(n_0))\cdot\langle 1\rangle) - 2^{-2n_0} = \beta(\bar{\delta}_2(n_0+1)) - 2^{-2n_0} \\
&< \beta((\bar{\delta}_2(m_0))\cdot\langle 0\rangle) + 2^{-2m_0} = \beta((\bar{\delta}_2(m_0))\cdot\langle 1\rangle - 2^{-2m_0} \\
&= \beta(\bar{\delta}_2(m_0+1)) - 2^{-2m_0} = \beta(\delta_2)(m_0) - 2^{-2m_0} \leqq \sigma * \beta(\delta_2) = f(\delta_2)
\end{aligned}$$

thus $$f(\delta_1) < f(\delta_2).$$

This completes the proof for $i = 2$.

PROOF FOR $i = 3$.

Let $\sigma\colon R^* \to R$.

$\xi = \langle x_0, \cdots, x_{n-1} \rangle \in R^*$ is a *σ-sequence* if $n = 0$ or $n > 0$ and

$$0 < x_i < k_i \sigma(\bar{\xi}(j)) \qquad 0 \leqq j < n.$$

σ is a *strategy for* I if for every σ-sequence ξ as above with $n > 0$

$$0 < \sigma(\xi) < l_{n-1} x_{n-1}$$

For $\xi \in R^*$, $\sigma\colon R^* \to R$ put $s(\xi) = 0$ if $|\xi| = 0$, and

$$s(\xi) = \sum_{i=0}^{|\xi|-1} (\sigma(\bar{\xi}(i)) + \xi(i))$$

for $|\xi| > 0$.

Fix a strategy σ for I.

$\alpha \in R^\omega$ is *good for* σ if $\bar{\alpha}(n)$ is a σ-sequence for all $n \in \omega$.

Let $\sigma * \alpha = \lim s(\bar{\alpha}(n))$ for α good for σ, when $s(\bar{\alpha}(n))$ is bounded, and

$$A_\sigma = \{\sigma * \alpha\colon \alpha \text{ is good for } \alpha \text{ and } s(\bar{\alpha}(n)) \text{ is bounded}\}$$

We show that A_σ includes a perfect set.

We define by induction on the length n of $u \in 2^*$

$$\beta\colon 2^* \to R$$

for short, put

$$\beta_u = \beta(u)$$

$$\tilde{\beta}_u = \tilde{\beta}(u)$$

$$\sigma_u = \sigma(\tilde{\beta}_u)$$

and:

$$s_u = s(\tilde{\beta}_u) + \sigma_u$$

Put

$$\beta(\emptyset) = 0; \text{ thus } \tilde{\beta}_\phi = \emptyset, \sigma_\phi = \sigma(\emptyset), s_\phi = \sigma(\emptyset)$$

For the induction step, put

$$\kappa_n = \min(1, k_0, \cdots, k_n)$$

$$\text{(11)} \qquad \lambda_n = \max(1, l_n)$$

$$a_n = \frac{1}{2} \frac{\kappa_n}{1 + l_n} \frac{1}{\lambda_n}$$

define

$$\beta_{u \cdot \langle 0 \rangle} = a_n^2 \sigma_u$$

$$\beta_{u \cdot \langle 1 \rangle} = a_n \sigma_u$$

Since by (11) $a_n^2 \leqq a_n \leqq \min(\frac{1}{2}, k_n)$ we have by induction: β_u is a σ-sequence for $u \in 2^*$.

The following inequalities hold

$$s_u < s_{u\cdot\langle 0\rangle} < s_{u\cdot\langle 0\rangle} + \sigma_{u\cdot\langle 0\rangle} < s_{u\cdot\langle 1\rangle} < s_{u\cdot\langle 1\rangle} + \sigma_{u\cdot\ 1} < s_u + \sigma_u. \tag{12}$$

Of these, only two are not obvious:

$$s_{u\cdot\langle 0\rangle} + \sigma_{u\cdot\langle 0\rangle} < s_{u\cdot\langle 1\rangle} \tag{13}$$

and

$$s_{u\cdot\langle 1\rangle} + \sigma_{u\cdot\langle 1\rangle} < s_u + \sigma_u \tag{14}$$

(13) is equivalent to:

$$(\beta_{u\cdot\langle 0\rangle} + \sigma_{u\cdot\langle 0\rangle}) + \sigma_{u\cdot\langle 0\rangle} < \beta_{u\cdot\langle 1\rangle} + \sigma_{u\cdot\langle 1\rangle}$$

We show that, moreover

$$(\beta_{u\cdot\langle 0\rangle} + \sigma_{u\cdot\langle 0\rangle}) + \sigma_{u\cdot\langle 0\rangle} < \beta_{u\cdot\langle 1\rangle}$$

By (11):

$$\begin{aligned}
&(\beta_{u\cdot\langle 0\rangle} + \sigma_{u\cdot\langle 0\rangle}) + \sigma_{u\cdot\langle 0\rangle} < (\beta_{u\cdot\langle 0\rangle} + l_n\beta_{u\cdot\langle 0\rangle}) + l_n\beta_{u\cdot\langle 0\rangle} \\
&= (1 + l_n)a_n^2\sigma_u + l_n a_n^2\sigma_u \\
&= \left[(1 + l_n)\cdot\frac{1}{4}\cdot\frac{\kappa_n^2}{(1 + l_n)^2}\cdot\ \frac{1}{\lambda_n^2} + l_n\cdot\frac{1}{4}\cdot\frac{\kappa_n^2}{(1 + l_n)^2}\cdot\frac{1}{\lambda_n^2}\right]\cdot\sigma_u \\
&\leqq \frac{1}{4}\cdot\frac{\kappa_n}{1 + l_n}\cdot\frac{1}{\lambda_n}\left[\frac{\kappa_n}{\lambda_n} + \frac{\kappa_n}{1 + l_n}\right]\cdot\sigma_u < 2\cdot\frac{1}{4}\cdot\frac{\kappa_n}{1 + l_n}\cdot\frac{1}{\lambda_n}\cdot\sigma_u \\
&= a_n\sigma_u = \beta_{u\cdot\langle 1\rangle}
\end{aligned}$$

and (13) is proved.

(14) is equivalent to

$$(\beta_{u\cdot\langle 1\rangle} + \sigma_{u\cdot\langle 1\rangle}) + \sigma_{u\cdot\langle 1\rangle} < \sigma_u.$$

We have, by (11)

$$(\beta_{u\cdot\langle 1\rangle} + \sigma_{u\cdot\langle 1\rangle}) + \sigma_{u\cdot\langle 1\rangle} < (\beta_{u,\langle 1\rangle} + l_n\beta_{u\cdot\langle 1\rangle}) + l_n\beta_{u\cdot\langle 1\rangle}$$

$$= \left[(1 + l_n)a_n + l_n a_n\right]\sigma_u = \left[(1 + l_n)\cdot\frac{1}{2}\cdot\frac{\kappa_n}{l + l_n}\cdot\frac{1}{\lambda_n} + l_n\cdot\frac{1}{2}\cdot\frac{\kappa_n}{1 + l_n}\cdot\frac{1}{\lambda_n}\right]\sigma_u$$

$$\leqq \frac{1}{2}\left[\frac{\kappa_n}{\lambda_n} + \frac{\kappa_n}{1 + l_n}\right]\sigma_u < \sigma_u$$

and (14) is proved.

Since β_u is a σ-sequence for $u \in 2^*$, we get that for $\delta \in 2^\omega$, $\beta(\delta)$ defined by

$$\beta(\delta)(n) = \beta(\bar{\delta}(n + 1))$$

is good for σ.

Put $P_u = [s_u, s_u + \sigma_u]$ for $u \in 2^*$. By (12), we have

$$P_{u\cdot\langle 0\rangle} \subseteq P_u, P_{u\cdot\langle 1\rangle} \subseteq P_u$$

and

$$P_{u\cdot\langle 0\rangle} \cap P_{u\cdot\langle 1\rangle} = \emptyset$$

So, by the lemma

$$P = \bigcap_{n<\omega} \bigcup_{|u|=n} P_u$$

is a perfect set.

Also, one sees by induction that $\sigma_u < 2^{-|u|}\sigma_{\emptyset}$, and P_u is a closed interval of length σ_u.

Hence, $x \in P$ iff for some $\delta \in 2^\omega$, $\{x\} = \bigcap_{n<\omega} P_{\bar{\delta}\langle n\rangle}$.

But this is true of x iff

$$x = \lim s_{\bar{\delta}(n)} = \lim s(\overline{\beta(\delta)}(n)),$$

Thus

$$P = \{\sigma * \beta(\delta) : \delta \in 2^\omega\} \subseteq A_\sigma.$$

This completes the proof of Theorem 1.

4. Proof of Theorem 2

We have to prove four statements:

(a) If S includes a perfect subset, then $\Gamma_1(S)$ is a win for I.

(b) If $\Gamma_1(S)$ is a win for I, then S includes a perfect set.

(c) If S is at most denumerable, then $\Gamma_1(S)$ is a win for II.

(d) If $\Gamma_1(S)$ is a win for II, then S is at most denumerable.

Of these, (a) and (c) were stated by Mycielski in [9], with no proof, so we sketch their proof for completeness.

It was pointed out to the author by J. Mycielski, that the proof of Theorem 1 for $i = 1$ is actually a proof of (b), and that R. Solovay proved it as well as (d) in 1968, but did not publish it.

(a) Assume that S includes a perfect subset P. We say that x has the property H if

H: for any $\varepsilon > 0$, $(x - \varepsilon, x) \cap P \neq \emptyset$ and $(x, x+\varepsilon) \cap P \neq \emptyset$.

Then, apart from an at most denumerable subset, every $x \in P$ has the property H. This follows from the fact, that we can assign to each $y \in P$, that does not satisfy H, an open interval disjoint from P with y as an end point in a manner that the intervals assigned to two different y's are disjoint. Since any family of pairwise disjoint intervals is countable, the set of such y's in P is also (at most) countable.

It follows that if $x \in P$ has the property H, and $\varepsilon > 0$, then one can always find another $x' \in P$ with the property H in the interval $(x-\varepsilon, x)$ or $(x, x+\varepsilon)$.

I wins $\Gamma_1(S)$ by choosing in his turn always an element of P with the property H. (Such a strategy can be described effectively).

(c) Assume that S is countable; say, $S = \{z_n : n < \omega\}$. II in his nth step defines the interval (x_{2n+1}, x_{2n}), in which the result of the game will be found.

If is clear that II can always so choose x_{2n+1}, that $z_n \notin (x_{2n+1}, x_{2n})$.

To prove (d), we start with some notations. g, h denote nonempty, open intervals. For $g = (a,b)$, $\|g\|$ denotes the length of g, i.e.:

$$\|g\| = b - a$$

$\bar{g}$ denotes the closure of g, i.e.

$$\bar{g} = [a,b]$$

and $l(g)(r(g))$ denotes the left (right) endpoint of g, i.e.

$$l(g) = a \quad (r(g) = b).$$

Let $g = (a,b)$ be a nonempty interval, $f: R \to R$ a function such that for $x \in g$, $f(x) < x$.

We associate with f and g a family of disjoint subintervals of g, denoted by $\Omega(f,g)$, with the following properties:

(1) $g - \bigcup \Omega(f,g)$ is denumerable.

(2) If $h \in \Omega(f,g)$, then $\|h\| \leqq \frac{1}{2}\|g\|$

(3) If $h \in \Omega(f,g)$, then $r(h) < r(g)$

$\Omega(f,g)$ is defined in three steps

1. Define

$$a_n = b - 2^{-n}\|g\|$$

$$(\text{i.e. } a_0 = a,\ a_1 = a + \tfrac{1}{2}(b-a),\ a_2 = a + \tfrac{3}{4}(b-a), \cdots)$$

then $a_n \in \bar{g}$ for all n. in fact, $a_n \in g$ for $0 < n$.

put $$g_n = (a_n, a_{n+1})$$

2. For $n < \omega$ define a well ordered continuous nonincreasing sequence x^n_ξ of points in $\bar{g}_n$ by

$$x^n_0 = a_{n+1}$$

$$x^n_{\xi+1} = \max(a_n, f(x^n_\xi))$$

and for a limit ordinal λ

$$x^n_\lambda = \inf \{x^n_\xi : \xi < \lambda\}$$

Put

$$\alpha_n = \min \{\gamma : x^n_\gamma = a_n\}$$

We have

(4) $\zeta < \xi < \alpha_n$ implies $x^n_\xi < x^n_\zeta$

(5) $\xi < \alpha_n$, $\zeta < \alpha_{n+1}$ implies $x^n_\xi < x^{n+1}_\zeta$

By (4), α_n is (at most) countable.

3. Define $\Omega(f,g)$ by

$h \in \Omega(f,g)$ iff there is an $n < \omega$ and a $\xi < \alpha_n$ such that $h = (x^n_{\xi+1}, x^n_\xi)$

By (4) and (5), $\Omega(f,g)$ is a family of disjoint subintervals of g.

It is easy to see that

$$g - \bigcup \Omega(f,g) = \{x^n_\xi : \xi < \alpha_n,\ n < \omega\}$$

So this set is countable, and (1) holds.
$h \in \Omega(f,g)$ implies $h \subseteq g_n$ for some n, and by construction

$$\|g_n\| = 2^{-(n+1)}\|g\|$$

so (2) holds,

For the same reason, $h \in \Omega(f,g)$ has always an n such that

$$r(h) \leqq r(g_n) = a_n < b = r(g)$$

so (3) holds.

Let $\tau : R^* \to R$.

$\xi = \langle x_0, \cdots, x_{n-1}\rangle \in R^*$ is a *τ-sequence* iff $1 \leqq n$ and

$$\tau(\langle x_0\rangle) < \cdots < \tau(\xi) < x_{n-1} < \cdots < x_0$$

τ is a *strategy for II* iff

(6) $\langle x_0\rangle$ is a τ-sequence, $x_0 \in R$

(7) If $\xi = \langle x_0, \cdots, x_{n-1}\rangle$ is a τ-sequence, $x \in R$, and

$$\tau(\xi) < x < x_{n-1}$$

then

$$\tau(\xi) < \tau(\xi \cdot \langle x\rangle) < x$$

$\alpha \in R^{\omega}$ is *good for* τ iff $\bar{\alpha}(n)$ is a τ-sequence for $1 < n$.

If τ is a strategy for II, α is good for τ, then α is a decreasing and bounded below (by $\tau(\langle\alpha(0)\rangle)$) sequence. In this case, define:

$$\alpha * \tau = \lim \alpha(n)$$

PROOF OF (d)

Assume that $\Gamma_1(S)$ is a win for II, and let τ be a winning strategy for II.

Put $B_\tau = \{\alpha * \tau \colon \alpha \text{ is good for } \tau\}$.

In order to show that S is at most denumerable, it is enough to find a $B \subseteq B_\tau$ such that $R - B$ is denurmerable, since, by definition of B_τ, $S \subseteq R - B_\tau \subseteq R - B$.

To this end, we define by induction a sequence Ω_n of families of open intervals so that:

(8) Ω_n is a family of pairwise disjoint nonempty open intervals, and $R = T_n \cup \cup \Omega_n$, where T_n is denumerable.

(9) $g \in \Omega_n$ implies $\| g \| \leqq 2^{-(n+1)}$

(10) Every $g \in \Omega_{n+1}$ has a $g' \in \Omega_n$ such that $g \subseteq g'$.

By (8), g' of (10) is unique. Thus, $g = (a,b) \in \Omega_n$ has a unique sequence $g = g_n \subseteq \cdots \subseteq g_1 \subseteq g_0$, $g_i = (a_i, b_i)$, such that $g_i \in \Omega_i$, $i = 0, \cdots, n$. We denote $\langle b_0, \cdots, b_n \rangle$ by $\xi(g)$. The extra property of the sequence Ω_n is then

(11) For $g \in \Omega_n$, $\xi(g)$ is a τ-sequence.

Assume that such a sequence Ω_n has already been defined. Denote:

$$G_n = \bigcup \Omega_n$$

and put

$$B = \bigcap_{n<\omega} G_n.$$

Then $x \notin B$ implies $x \in \bigcup_{n<\omega} T_n$, by (8), so B has a denumerable complement.

On the other hand, if $x \in B$, then by (8), for every $n \in \omega$ there is a unique $g_n = (a_n, b_n)$ in Ω_n such that $x \in g_n$. By (9), $\{x\} = \cap_n \bar{g}_n$, so that $x = \lim b_n$ Put $\alpha(n) = b_n$. By (11), α is good for τ, and we conclude

$$x = \alpha * \tau \in B_\tau$$

so

$$B \subseteq B_\tau.$$

It remains to define by induction the family Ω_n.

Z denotes the set of the integers. Put $f_0(x) = \tau(\langle x \rangle)$, $x \in R$, then $f_0(x) < x$ for all x.

$$\Omega_0 = \bigcup \{\Omega(f_0, (n, n+1)) \colon n \in Z\}$$

(8) holds by (1). (9) holds by (2). (10) holds trivially. (11) holds by (6).

We also have: if $g \in \Omega_0$, then

$$\tau(\xi(g)) \leqq l(g) \tag{12}$$

(see 2. in the definition of $\Omega(f, g)$).

Assume that $\Omega_0, \cdots, \Omega_n$ are defined and that (8)–(12) hold.

With $g \in \Omega_n$ associate f_g defined by:

$$f_g(x) = \tau((\xi(g)) \cdot \langle x \rangle)$$

By (12) and (7) we have: if $x \in g$, then $f_g(x) < x$. Define

$$\Omega_{n+1} = \bigcup \{\Omega(f_g, g) \colon g \in \Omega_n\}$$

(8) for Ω_n implies that Ω_n is denumerable. This with (1) gives (8) for Ω_{n+1}.

(9) follows by (2), and the induction hypothesis.

(10) follows by the appropriate property of Ω.

(11) is a consequence of (11) and (12) for Ω_n.

(12) follows again by the property of Ω, described in 2. of its definition.

This completes the proof of Theorem 2.

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